Complex Discrete Fourier Transform

By José Luis Gómez-Muñoz
http://homepage.cem.itesm.mx/jose.luis.gomez/

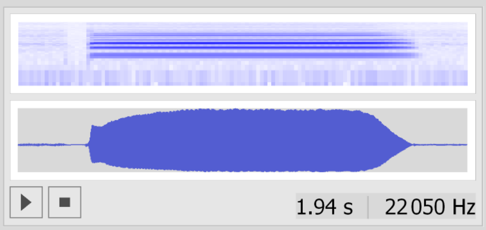
Oboe Recording

This is an Oboe recording (a WAV file) included in every computer that has *Mathematica*. Below the sound is stored in the variable “recording” (if you are reading this document in *Mathematica* or the *CDFPlayer*, press the button ▶ in the result of the calculation):

In[1]:=

```
recording = ExampleData[{"Sound", "Oboe"}]
```

Out[1]=



Numerical Data of the Oboe Recording

The recording is actually a large list of numbers, below they are stored in the variable “data”:

In[2]:=

```
data = ExampleData[{"Sound", "Oboe"}, "Data"]
```

Out[2]=

A very large output was generated. Here is a sample of it:

```
{{0.00305185, 0.0052797, 0.00704978,
  0.00732444, 0.00613422, 0.00653096, <<42 842>>, 0.00454726,
  0.0029603, 0.00125126, 0.00216681, 0.00543229, 0.004944}}
```

Show Less | Show More | Show Full Output | Set Size Limit...

The sample rate means how many samples from the previous list correspond to a time of one second, below it is stored in the variable “samplerate”:

In[3]:=

```
samplerate = ExampleData[{"Sound", "Oboe"}, "SampleRate"]
```

Out[3]=

```
22 050
```

The number of channels (1=mono, 2=stereo, etc) and the number of samples (length of the numerical data) of our recording are stored below in the variables “nChannel” and “nSamples”:

In[4]:=
```
{nchannels, nsamples} = Dimensions[data]
```
Out[4]=
```
{1, 42 854}
```

Three Milliseconds of Waveform

The data of the first channel is stored below in a variable:

In[5]:=
```
firstchanneldata = Part[data, 1]
```
Out[5]=

A very large output was generated. Here is a sample of it:

```
{0.00305185, 0.0052797, 0.00704978, 0.00732444,
 0.00613422, 0.00653096, <<42 842>>, 0.00454726,
 0.0029603, 0.00125126, 0.00216681, 0.00543229, 0.004944}
```

Show Less | Show More | Show Full Output | Set Size Limit...

Below we obtain the sample number corresponding to the middle of the data:

In[6]:=
```
middleofrecording = IntegerPart[nsamples/2]
```
Out[6]=
```
21 427
```

Below we obtain the number of samples corresponding to three milliseconds

In[7]:=
```
threemilliseconds = Ceiling[0.003 * samplerate]
```
Out[7]=
```
67
```

Below three milliseconds of data are extracted from the middle of the data, and they are stored in a variable:

In[8]:=

```
threemsdata =
 Part[firstchanneldata,
  Span[middleofrecording, (middleofrecording + threemilliseconds)]]
```

Out[8]=

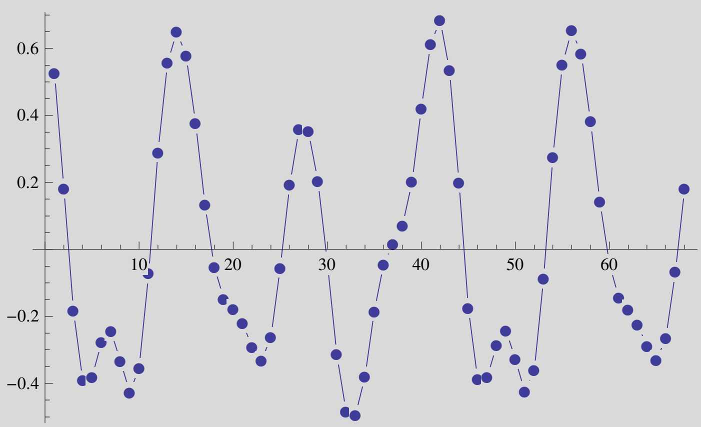
```
{0.525163, 0.181646, -0.183233, -0.391247, -0.380932, -0.276772, -0.244057,
 -0.33375, -0.427015, -0.354167, -0.0721152, 0.28956, 0.557054, 0.649739,
 0.578875, 0.375958, 0.134098, -0.0522172, -0.147862, -0.178533, -0.22071,
 -0.291696, -0.333018, -0.262093, -0.0550554, 0.193823, 0.359569, 0.353191,
 0.20365, -0.0358287, -0.313791, -0.48442, -0.495346, -0.3802, -0.185186,
 -0.0450758, 0.0163274, 0.0703757, 0.201422, 0.420392, 0.612781, 0.684866,
 0.535356, 0.199469, -0.175817, -0.386944, -0.381481, -0.286966, -0.2425,
 -0.328257, -0.424146, -0.361583, -0.0878933, 0.274728, 0.552232,
 0.654134, 0.584155, 0.381695, 0.142491, -0.0438551, -0.143925, -0.180395,
 -0.224372, -0.288766, -0.330454, -0.265542, -0.0671712, 0.181555}
```

Three milliseconds of waveform are ploted below:

In[9]:=

```
ListLinePlot[threemsdata, PlotMarkers → Automatic]
```

Out[9]=



Complex Discrete Fourier Transform

The Discrete Fourier Transform (DFT), which is usually calculated using the Fast Fourier Transform (FFT) algorithm, is a list of complex numbers, as shown below:

In[10]:=

```
discretefourier = Fourier[threemsdata]
```

Out[10]=

```
{-0.0545811 + 0. i, -0.0945907 - 0.088914 i, -0.270561 + 0.0758201 i,
 -0.461966 - 0.529857 i, -0.261522 + 0.314779 i, 1.59073 - 0.10004 i,
 0.182306 + 0.227261 i, 0.025641 - 0.135517 i, -0.00887291 - 0.0330058 i,
 -0.199889 - 0.0153932 i, 0.645946 - 0.0527299 i, 0.203777 - 0.150993 i,
 0.0883988 + 0.0784085 i, 0.154806 + 0.0294135 i, 0.0987279 + 0.0315506 i,
 0.0111128 + 0.0173925 i, 0.0336593 + 0.0190111 i, 0.0339633 + 0.0105365 i,
 0.0361977 + 0.00910133 i, 0.0383419 + 0.004289 i, 0.0221039 + 0.0133263 i,
 0.0311801 + 0.00780741 i, 0.0240568 + 0.00918186 i, 0.0248504 + 0.00220147 i,
 0.0279029 + 0.00803028 i, 0.0216736 + 0.00222649 i, 0.024363 + 0.00671732 i,
 0.0195836 + 0.00467345 i, 0.0283328 + 0.00132894 i, 0.0203762 + 0.00288926 i,
 0.0247652 + 0.00315915 i, 0.0223059 - 0.00119374 i, 0.0211916 + 0.00124362 i,
 0.0225228 + 0.000245662 i, 0.0223461 + 0. i, 0.0225228 - 0.000245662 i,
 0.0211916 - 0.00124362 i, 0.0223059 + 0.00119374 i,
 0.0247652 - 0.00315915 i, 0.0203762 - 0.00288926 i, 0.0283328 - 0.00132894 i,
 0.0195836 - 0.00467345 i, 0.024363 - 0.00671732 i, 0.0216736 - 0.00222649 i,
 0.0279029 - 0.00803028 i, 0.0248504 - 0.00220147 i, 0.0240568 - 0.00918186 i,
 0.0311801 - 0.00780741 i, 0.0221039 - 0.0133263 i, 0.0383419 - 0.004289 i,
 0.0361977 - 0.00910133 i, 0.0339633 - 0.0105365 i, 0.0336593 - 0.0190111 i,
 0.0111128 - 0.0173925 i, 0.0987279 - 0.0315506 i, 0.154806 - 0.0294135 i,
 0.0883988 - 0.0784085 i, 0.203777 + 0.150993 i, 0.645946 + 0.0527299 i,
 -0.199889 + 0.0153932 i, -0.00887291 + 0.0330058 i, 0.025641 + 0.135517 i,
 0.182306 - 0.227261 i, 1.59073 + 0.10004 i, -0.261522 - 0.314779 i,
 -0.461966 + 0.529857 i, -0.270561 - 0.0758201 i, -0.0945907 + 0.088914 i}
```

The Waveform as a Sum of Complex Exponentials

The original waveform is a sum of complex exponentials multiplied by the complex numbers of the Discrete Fourier Transform, as shown below:

In[11]:=

```
Clear[k];
n = Length[discretefourier];
fourierwaveform =
```

$$\frac{1}{\sqrt{n}} \sum_{m=1}^{n} (\text{discretefourier}[[m]] * \text{Exp}[-2 * \pi * i * (m-1) * (k-1) / n])$$

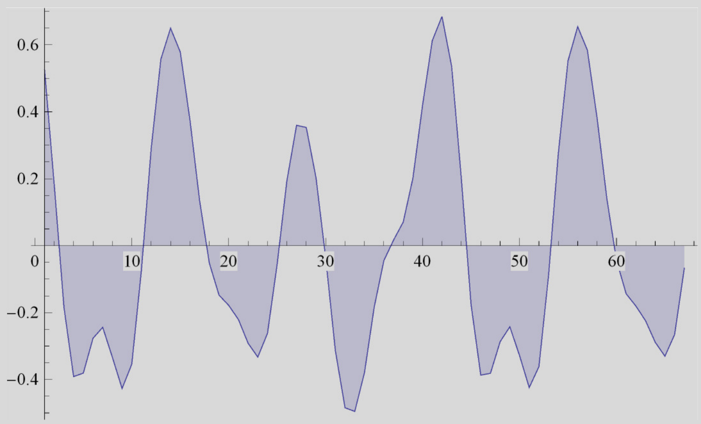
Out[13]=

$$
\begin{aligned}
&\frac{1}{2\sqrt{17}}\Big((-0.0545811+0.\,i)-(0.0945907+0.088914\,i)\,e^{-\frac{1}{34}i(-1+k)\pi}-\\
&(0.270561-0.0758201\,i)\,e^{-\frac{1}{17}i(-1+k)\pi}-(0.461966+0.529857\,i)\,e^{-\frac{3}{34}i(-1+k)\pi}-\\
&(0.261522-0.314779\,i)\,e^{-\frac{2}{17}i(-1+k)\pi}+(1.59073-0.10004\,i)\,e^{-\frac{5}{34}i(-1+k)\pi}+\\
&(0.182306+0.227261\,i)\,e^{-\frac{3}{17}i(-1+k)\pi}+(0.025641-0.135517\,i)\,e^{-\frac{7}{34}i(-1+k)\pi}-\\
&(0.00887291+0.0330058\,i)\,e^{-\frac{4}{17}i(-1+k)\pi}-(0.199889+0.0153932\,i)\,e^{-\frac{9}{34}i(-1+k)\pi}+\\
&(0.645946-0.0527299\,i)\,e^{-\frac{5}{17}i(-1+k)\pi}+(0.203777-0.150993\,i)\,e^{-\frac{11}{34}i(-1+k)\pi}+\\
&(0.0883988+0.0784085\,i)\,e^{-\frac{6}{17}i(-1+k)\pi}+(0.154806+0.0294135\,i)\,e^{-\frac{13}{34}i(-1+k)\pi}+\\
&(0.0987279+0.0315506\,i)\,e^{-\frac{7}{17}i(-1+k)\pi}+(0.0111128+0.0173925\,i)\,e^{-\frac{15}{34}i(-1+k)\pi}+\\
&(0.0336593+0.0190111\,i)\,e^{-\frac{8}{17}i(-1+k)\pi}+(0.0339633+0.0105365\,i)\,e^{-\frac{1}{2}i(-1+k)\pi}+\\
&(0.0361977+0.00910133\,i)\,e^{-\frac{9}{17}i(-1+k)\pi}+(0.0383419+0.004289\,i)\,e^{-\frac{19}{34}i(-1+k)\pi}+\\
&(0.0221039+0.0133263\,i)\,e^{-\frac{10}{17}i(-1+k)\pi}+(0.0311801+0.00780741\,i)\,e^{-\frac{21}{34}i(-1+k)\pi}+\\
&(0.0240568+0.00918186\,i)\,e^{-\frac{11}{17}i(-1+k)\pi}+(0.0248504+0.00220147\,i)\,e^{-\frac{23}{34}i(-1+k)\pi}+\\
&(0.0279029+0.00803028\,i)\,e^{-\frac{12}{17}i(-1+k)\pi}+(0.0216736+0.00222649\,i)\,e^{-\frac{25}{34}i(-1+k)\pi}+\\
&(0.024363+0.00671732\,i)\,e^{-\frac{13}{17}i(-1+k)\pi}+(0.0195836+0.00467345\,i)\,e^{-\frac{27}{34}i(-1+k)\pi}+\\
&(0.0283328+0.00132894\,i)\,e^{-\frac{14}{17}i(-1+k)\pi}+(0.0203762+0.00288926\,i)\,e^{-\frac{29}{34}i(-1+k)\pi}+\\
&(0.0247652+0.00315915\,i)\,e^{-\frac{15}{17}i(-1+k)\pi}+(0.0223059-0.00119374\,i)\,e^{-\frac{31}{34}i(-1+k)\pi}+\\
&(0.0211916+0.00124362\,i)\,e^{-\frac{16}{17}i(-1+k)\pi}+(0.0225228+0.000245662\,i)\,e^{-\frac{33}{34}i(-1+k)\pi}+\\
&(0.0223461+0.\,i)\,e^{-i(-1+k)\pi}+(0.0225228-0.000245662\,i)\,e^{-\frac{35}{34}i(-1+k)\pi}+\\
&(0.0211916-0.00124362\,i)\,e^{-\frac{18}{17}i(-1+k)\pi}+(0.0223059+0.00119374\,i)\,e^{-\frac{37}{34}i(-1+k)\pi}+\\
&(0.0247652-0.00315915\,i)\,e^{-\frac{19}{17}i(-1+k)\pi}+(0.0203762-0.00288926\,i)\,e^{-\frac{39}{34}i(-1+k)\pi}+\\
&(0.0283328-0.00132894\,i)\,e^{-\frac{20}{17}i(-1+k)\pi}+(0.0195836-0.00467345\,i)\,e^{-\frac{41}{34}i(-1+k)\pi}+\\
&(0.024363-0.00671732\,i)\,e^{-\frac{21}{17}i(-1+k)\pi}+(0.0216736-0.00222649\,i)\,e^{-\frac{43}{34}i(-1+k)\pi}+\\
&(0.0279029-0.00803028\,i)\,e^{-\frac{22}{17}i(-1+k)\pi}+(0.0248504-0.00220147\,i)\,e^{-\frac{45}{34}i(-1+k)\pi}+\\
&(0.0240568-0.00918186\,i)\,e^{-\frac{23}{17}i(-1+k)\pi}+(0.0311801-0.00780741\,i)\,e^{-\frac{47}{34}i(-1+k)\pi}+\\
&(0.0221039-0.0133263\,i)\,e^{-\frac{24}{17}i(-1+k)\pi}+(0.0383419-0.004289\,i)\,e^{-\frac{49}{34}i(-1+k)\pi}+\\
&(0.0361977-0.00910133\,i)\,e^{-\frac{25}{17}i(-1+k)\pi}+(0.0339633-0.0105365\,i)\,e^{-\frac{3}{2}i(-1+k)\pi}+\\
&(0.0336593-0.0190111\,i)\,e^{-\frac{26}{17}i(-1+k)\pi}+(0.0111128-0.0173925\,i)\,e^{-\frac{53}{34}i(-1+k)\pi}+\\
&(0.0987279-0.0315506\,i)\,e^{-\frac{27}{17}i(-1+k)\pi}+(0.154806-0.0294135\,i)\,e^{-\frac{55}{34}i(-1+k)\pi}+\\
&(0.0883988-0.0784085\,i)\,e^{-\frac{28}{17}i(-1+k)\pi}+(0.203777+0.150993\,i)\,e^{-\frac{57}{34}i(-1+k)\pi}+\\
&(0.645946+0.0527299\,i)\,e^{-\frac{29}{17}i(-1+k)\pi}-(0.199889-0.0153932\,i)\,e^{-\frac{59}{34}i(-1+k)\pi}-\\
&(0.00887291-0.0330058\,i)\,e^{-\frac{30}{17}i(-1+k)\pi}+(0.025641+0.135517\,i)\,e^{-\frac{61}{34}i(-1+k)\pi}+\\
&(0.182306-0.227261\,i)\,e^{-\frac{31}{17}i(-1+k)\pi}+(1.59073+0.10004\,i)\,e^{-\frac{63}{34}i(-1+k)\pi}-\\
&(0.261522+0.314779\,i)\,e^{-\frac{32}{17}i(-1+k)\pi}-(0.461966-0.529857\,i)\,e^{-\frac{65}{34}i(-1+k)\pi}-\\
&(0.270561+0.0758201\,i)\,e^{-\frac{33}{17}i(-1+k)\pi}-(0.0945907-0.088914\,i)\,e^{-\frac{67}{34}i(-1+k)\pi}\Big)
\end{aligned}
$$

Below we plot the sum of complex exponentials, using DiscretePlot so that the variable k only takes integer values, and using Chop so that imaginary parts that are very close to zero are replaced with an exact zero:

In[14]:=
```
Clear[k];
DiscretePlot[Chop[fourierwaveform], {k, 1, threemilliseconds}, Joined → True]
```

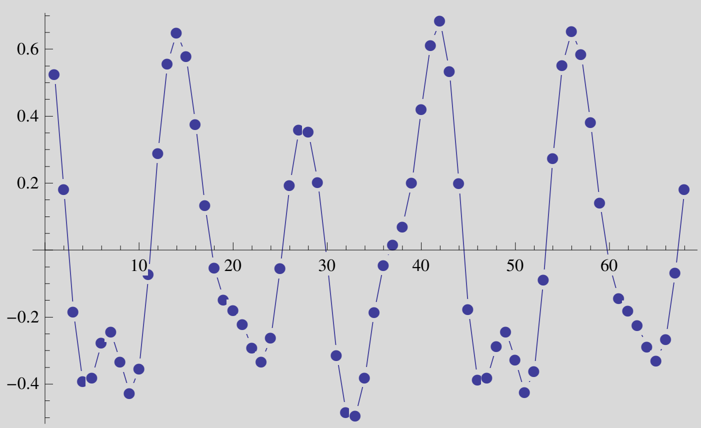
Out[15]=



Compare the waveform made of the sum of complex exponentials above, with the original waveform from the recording (a WAV file), which is shown again below:

In[16]:=
```
ListLinePlot[threemsdata, PlotMarkers → Automatic]
```

Out[16]=



Exercises

Exercise 1

Select, from the list below, one of the recordings of a **wind** or **string** instrument that are included in *Mathematica*, and obtain the Discrete Fourier Transform (DFT) of three milliseconds at the middle of the recording;

In[17]:=
```
ExampleData["Sound"]
```

Out[17]=
```
{{Sound, AltoFlute}, {Sound, AltoFluteScale}, {Sound, AltoSaxophone},
 {Sound, AltoSaxophoneScale}, {Sound, Apollo11PhoneCall},
 {Sound, Apollo11ReturnSafely}, {Sound, Apollo11SmallStep},
 {Sound, Apollo13Countdown}, {Sound, Apollo13Problem}, {Sound, BalloonPop},
 {Sound, BassClarinet}, {Sound, BassClarinetScale}, {Sound, BassFlute},
 {Sound, BassFluteScale}, {Sound, Bassoon}, {Sound, BassoonScale},
 {Sound, BassTrombone}, {Sound, BassTromboneScale}, {Sound, BlackcapWarbler},
 {Sound, Burst100}, {Sound, Burst1000}, {Sound, Burst7350},
 {Sound, Cello}, {Sound, CelloPizzicato}, {Sound, CelloPizzicatoScale},
 {Sound, CelloScale}, {Sound, Clarinet}, {Sound, ClarinetScale},
 {Sound, CrashCymbal}, {Sound, DoubleBass}, {Sound, DoubleBassPizzicato},
 {Sound, DoubleBassPizzicatoScale}, {Sound, DoubleBassScale}, {Sound, Flute},
 {Sound, FluteScale}, {Sound, FrenchHorn}, {Sound, FrenchHornScale},
 {Sound, JetSound}, {Sound, LinearSweep}, {Sound, NoiseBlue},
 {Sound, NoiseBrown}, {Sound, NoisePink}, {Sound, NoiseViolet},
 {Sound, NoiseWhite}, {Sound, Oboe}, {Sound, OboeScale}, {Sound, OrganChord},
 {Sound, Piano}, {Sound, PianoScale}, {Sound, RollingCoin},
 {Sound, SopranoSaxophone}, {Sound, SopranoSaxophoneScale},
 {Sound, Square10}, {Sound, Square100}, {Sound, Square1000},
 {Sound, Square7350}, {Sound, SubwayTrain}, {Sound, TenorTrombone},
 {Sound, TenorTromboneScale}, {Sound, TestIntermodulationDistortion},
 {Sound, Trumpet}, {Sound, TrumpetScale}, {Sound, Tuba}, {Sound, TubaScale},
 {Sound, Viola}, {Sound, ViolaScale}, {Sound, Violin}, {Sound, ViolinScale}}
```

Exercise 2

Use the DFT from the previous exercise to obtain the expression of the waveform as a sum of complex exponentials.

Exercise 3

Plot the sum of complex exponentials from the previous exercise, and compare with a plot of the original waveform. They must be identical.

Author: José Luis Gómez-Muñoz http://homepage.cem.itesm.mx/jose.luis.gomez/

In[18]:=
```
{DateString[], $Version}
```

Out[18]=
```
{Fri 27 Feb 2015 12:15:51,
 9.0 for Microsoft Windows (64-bit) (January 25, 2013)}
```