Fourier Analysis

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Sigma Notation for Sums

Below we calculate the sum of the squares of the first five natural numbers:

In[1]:=
$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$
Out[1]=
55

Below we calculate the same sum using sigma notation:

In[2]:=
$$\sum_{n=1}^{5} n^2$$
Out[2]=
55

One of the advantages of the sigma notation is that we can easily write large sums. Below we calculate the sum of the squares of the first 500 natural numbers:

In[3]:=
$$\sum_{n=1}^{500} n^2$$
Out[3]=
41 791 750

The summation index is “dummy”, that means that if you change everywhere (below the sigma symbol, and in the expression) then you get the same result, compare the sum below with the previuos sum

In[4]:=
$$\sum_{k=1}^{500} k^2$$
Out[4]=
41 791 750

We can use the sigma notation to generate a large function, as shown below:

In[5]:=

$$\sum_{k=1}^{6}\left(\frac{1}{k}\,\text{Sin}[2*k*\text{Pi}*440*t]\right)$$

Out[5]=

$$\text{Sin}[880\,\pi\,t]+\frac{1}{2}\,\text{Sin}[1760\,\pi\,t]+\frac{1}{3}\,\text{Sin}[2640\,\pi\,t]+\frac{1}{4}\,\text{Sin}[3520\,\pi\,t]+\frac{1}{5}\,\text{Sin}[4400\,\pi\,t]+\frac{1}{6}\,\text{Sin}[5280\,\pi\,t]$$

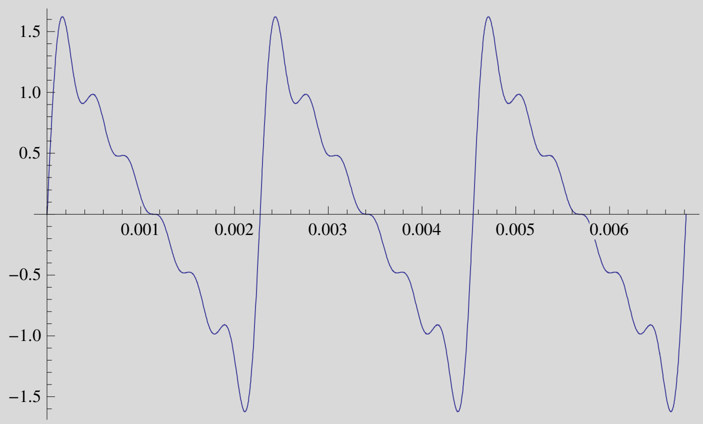
The function generated with sigma notation can be easily visualized, as shown below:

In[6]:=

$$\text{funct}=\sum_{k=1}^{6}\left(\frac{1}{k}\,\text{Sin}[2*k*\text{Pi}*440*t]\right);$$

$$\text{Plot}\left[\text{funct},\left\{t,0,\frac{3}{440}\right\}\right]$$

Out[7]=



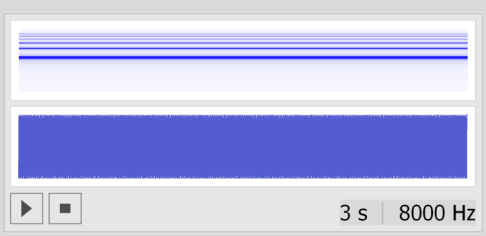
We can listen the function generated with sigma notation, as shown below:

In[8]:=

$$\text{funct}=\sum_{k=1}^{6}\left(\frac{1}{k}\,\text{Sin}[2*k*\text{Pi}*440*t]\right);$$

```
Play[funct, {t, 0, 3}]
```

Out[9]=



Below we have another example:

In[10]:=

$$\sum_{k=1}^{6}\left(\frac{1}{2\,k-1}\,\mathrm{Sin}[2*(2\,k-1)*\mathrm{Pi}*523*t]\right)$$

Out[10]=

$$\mathrm{Sin}[1046\,\pi\,t]+\frac{1}{3}\,\mathrm{Sin}[3138\,\pi\,t]+\frac{1}{5}\,\mathrm{Sin}[5230\,\pi\,t]+\frac{1}{7}\,\mathrm{Sin}[7322\,\pi\,t]+\frac{1}{9}\,\mathrm{Sin}[9414\,\pi\,t]+\frac{1}{11}\,\mathrm{Sin}[11\,506\,\pi\,t]$$

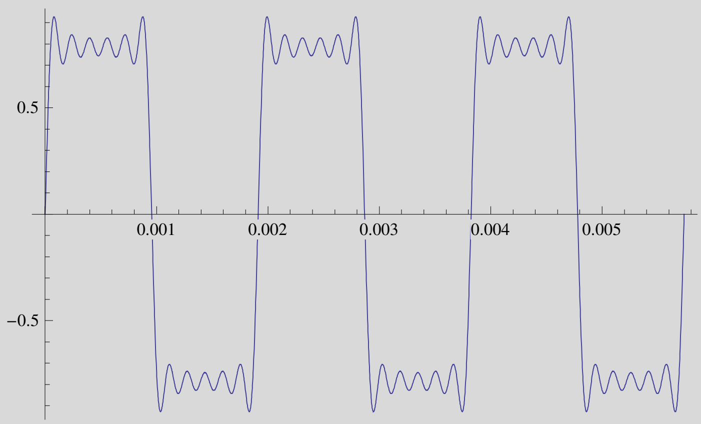
The function generated with sigma notation can be easily visualized, as shown below:

In[11]:=

$$\mathrm{funct}=\sum_{k=1}^{6}\left(\frac{1}{2\,k-1}\,\mathrm{Sin}[2*(2\,k-1)*\mathrm{Pi}*523*t]\right);$$

$$\mathrm{Plot}\left[\mathrm{funct},\left\{t,0,\frac{3}{523}\right\}\right]$$

Out[12]=



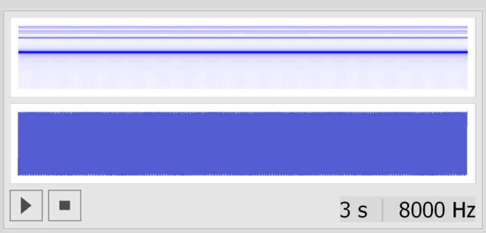
We can listen the function generated with sigma notation, as shown below:

In[13]:=

$$\mathrm{funct}=\sum_{k=1}^{6}\left(\frac{1}{2\,k-1}\,\mathrm{Sin}[2*(2\,k-1)*\mathrm{Pi}*523*t]\right);$$

```
Play[funct, {t, 0, 3}]
```

Out[14]=



Exercises

Exercise 1

Use the **sigma notation** to generate this wavefunction:

$$f(t) = \frac{1}{4}\sin(784\,\pi\,t) + \frac{1}{9}\sin(1568\,\pi\,t) + \frac{1}{16}\sin(2352\,\pi\,t) + \frac{1}{25}\sin(3136\,\pi\,t) + \frac{1}{36}\sin(3920\,\pi\,t) + \frac{1}{49}\sin(4704\,\pi\,t) + \frac{1}{64}\sin(5488\,\pi\,t) + \frac{1}{81}\sin(6272\,\pi\,t) + \frac{1}{100}\sin(7056\,\pi\,t) + \frac{1}{121}\sin(7840\,\pi\,t) + \frac{1}{144}\sin(8624\,\pi\,t) + \frac{1}{169}\sin(9408\,\pi\,t) + \frac{1}{196}\sin(10\,192\,\pi\,t) + \frac{1}{225}\sin(10\,976\,\pi\,t) + \frac{1}{256}\sin(11\,760\,\pi\,t) + \frac{1}{289}\sin(12\,544\,\pi\,t) + \frac{1}{324}\sin(13\,328\,\pi\,t) + \frac{1}{361}\sin(14\,112\,\pi\,t) + \frac{1}{400}\sin(14\,896\,\pi\,t) + \frac{1}{441}\sin(15\,680\,\pi\,t) + \frac{1}{484}\sin(16\,464\,\pi\,t) + \frac{1}{529}\sin(17\,248\,\pi\,t) + \frac{1}{576}\sin(18\,032\,\pi\,t) + \frac{1}{625}\sin(18\,816\,\pi\,t) + \frac{1}{676}\sin(19\,600\,\pi\,t) + \frac{1}{729}\sin(20\,384\,\pi\,t) + \frac{1}{784}\sin(21\,168\,\pi\,t) + \frac{1}{841}\sin(21\,952\,\pi\,t) + \frac{1}{900}\sin(22\,736\,\pi\,t) + \frac{1}{961}\sin(23\,520\,\pi\,t)$$

Exercise 2

Use *Mathematica* to graph **exactly three periods** of the wavefunction in the previous exercise

Exercise 3

Use *Mathematica* to listen to **exactly three seconds** of the wavefunction in the previous exercise

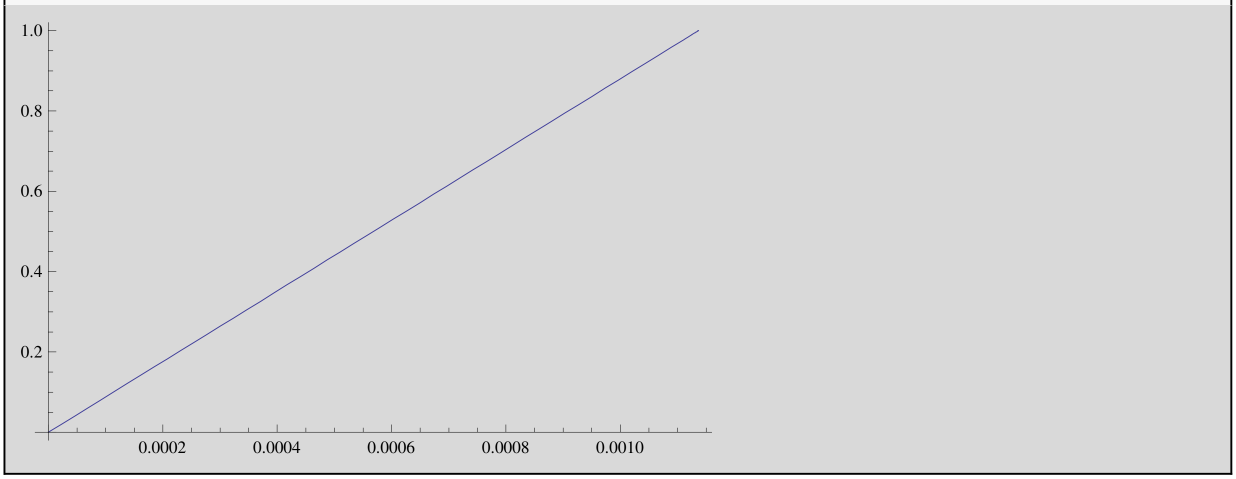
Fourier Sine Series

This is a graph of a linear function that goes from zero to one in an interval form $t = 0$ to $t = \frac{1}{880}$

In[15]:=

```
Plot[880 t, {t, 0, 1/880}]
```

Out[15]=



The command FourierSinSeries calculates the superposition of sine waves that approximates the function and makes it periodical:

In[16]:=

```
frequency = 440;
FourierSinSeries[880 t,
 t, 9, FourierParameters → {0, 2 * π * frequency}]
```

Out[17]=

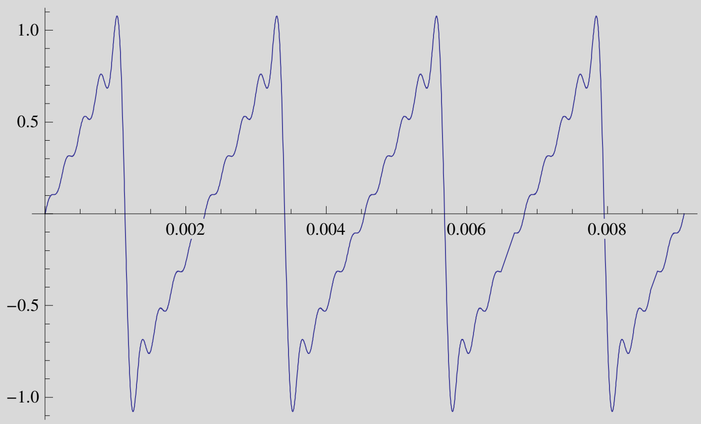
$$\frac{2\,\mathrm{Sin}[880\,\pi\,t]}{\pi} - \frac{\mathrm{Sin}[1760\,\pi\,t]}{\pi} + \frac{2\,\mathrm{Sin}[2640\,\pi\,t]}{3\,\pi} - \frac{\mathrm{Sin}[3520\,\pi\,t]}{2\,\pi} + \frac{2\,\mathrm{Sin}[4400\,\pi\,t]}{5\,\pi} - \frac{\mathrm{Sin}[5280\,\pi\,t]}{3\,\pi} + \frac{2\,\mathrm{Sin}[6160\,\pi\,t]}{7\,\pi} - \frac{\mathrm{Sin}[7040\,\pi\,t]}{4\,\pi} + \frac{2\,\mathrm{Sin}[7920\,\pi\,t]}{9\,\pi}$$

Below we plot four periods of the periodical approximation to the function:

In[18]:=

```
frequency = 440;
myseries = FourierSinSeries[880 t,
   t, 9, FourierParameters → {0, 2 * π * frequency}];
Plot[myseries, {t, 0, 4/frequency}]
```

Out[20]=

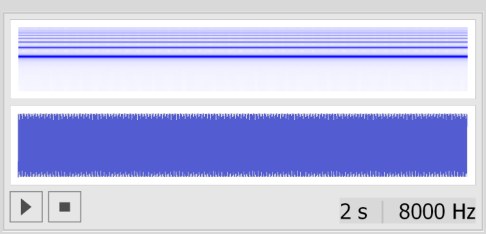


Below we play two seconds of the periodical approximation to the function:

In[21]:=

```
frequency = 440;
myseries = FourierSinSeries[880 t,
   t, 9, FourierParameters → {0, 2 * π * frequency}];
Play[myseries, {t, 0, 2}]
```

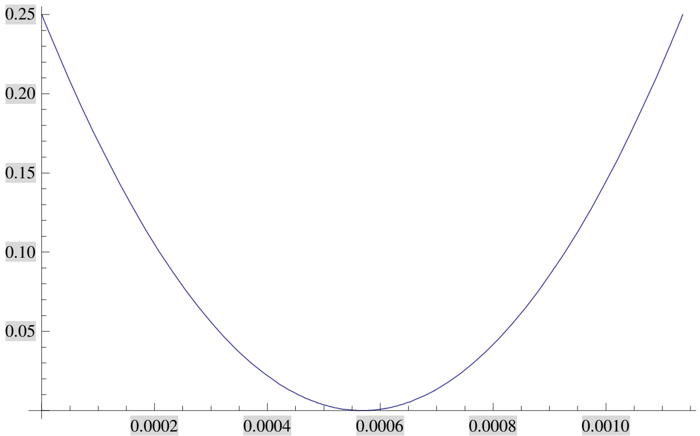
Out[23]=



Exercises

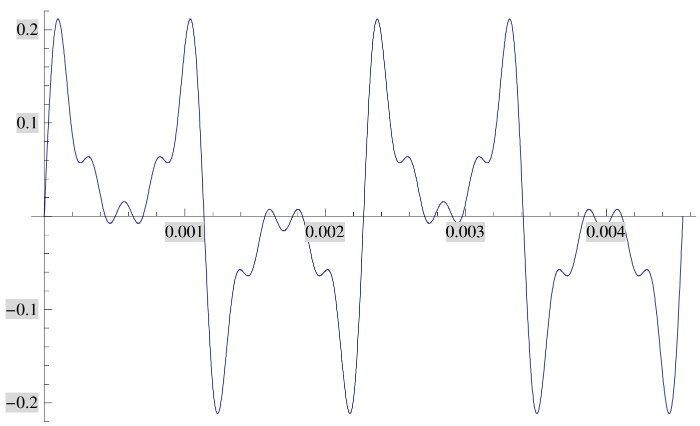
Exercise 4

Obtain a function that has exactly this graph:



Exercise 5

Use the FourierSinSeries command to make periodic the previous function, and graph two periods:



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In[24]:=

```
{DateString[], $Version}
```

Out[24]=

```
{Tue 24 Feb 2015 12:55:25,
 9.0 for Microsoft Windows (64-bit) (January 25, 2013)}
```